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Finite Element Analysis of Mooring Lines for Navigational Buoys with Attached Weights under Random Loads

Van Tuan Dao* and **Thi Diem Chi Nguyen**

Abstract: This paper presents a methodology for determining the wave surface profile and the velocity components of water particles from the wave spectrum of a random sea state to calculate the loads acting on the buoy. For the calculation of mooring lines with attached weights under random loads, the finite element method (FEM) is applied. The parameters of the governing dynamic equations are identified, the Newmark method is used for time integration, and a computational program is developed. The program is verified by comparing the results of the static problem with the dynamic problem under constant loading, demonstrating the correctness of the algorithm and computation. The program is then applied to calculate mooring lines with attached weights for an actual navigational buoy subjected to wind, current, and random wave loads, demonstrating the practical applicability of the proposed approach.

Keywords: Mooring line, vibration, random wave, wave spectrum, attached weight, random load.

1. INTRODUCTION

Navigational buoys are typically maintained in position by a single mooring line. To reduce displacements, attached weights are commonly placed along the mooring line. The mooring line is subjected to its self-weight and the environmental loads acting on the buoy due to wind, current, and waves. For buoys deployed offshore, the wave load is a random variable. In the current practice of designing mooring lines with attached weights, approximate formulas are frequently employed to estimate tension and displacement, while the loads acting on the buoy are generally considered as static. Examples include the U.S. Navy mooring design manual [4] and Russian design instructions [8].

Internationally, studies on mooring lines with suspended masses are limited. Notable research includes works by I.E. Udoh [5], E. Coarita and L.Flores [6], where the analyses were conducted for mooring systems without attached weights, and the loads were static or deterministic.

In Vietnam, N.Q. Hoa [2] calculated a single mooring line with suspended weights in a manner similar to the U.S. Navy guidelines. D.V. Tuan [3] analyzed mooring lines with attached weights but only under static loading. Other domestic studies have focused on mooring lines without suspended weights. To date, no published study has analyzed mooring lines with attached weights subjected to random wave loading using the finite element method (FEM). Generally, current calculations of mooring lines with attached weights still rely on single mooring line models with static loading, which is not comprehensive. Applying FEM to mooring lines with attached weights under random wave loading enables analyses that more accurately reflect actual structural behavior under environmental conditions.

Motivated by the above-mentioned review, we focus on finite element analysis of mooring lines for navigational buoys with attached weights influenced by random loads. The study holds the following contributions:

(i) For the general case, we analyze the nonlinear dynamics of mooring lines with an arbitrary number of suspended masses. We consider a sea state where the wave disturbance acting on the system is random and can have any spectrum.

(ii) We develop the algorithm and provide code for numerical simulation of mooring lines, considering influences close to real conditions. The results demonstrate the applicability of this approach to practical systems.

The article is structured in five sections. After

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Van Tuan Dao and Thi Diem Chi Nguyen are with the Faculty of Civil Engineering, Vietnam Maritime University, Haiphong, Vietnam (e-mails: tuandv.ctt@vimaru.edu.vn and chintd@vimaru.edu.vn).
* Corresponding author.

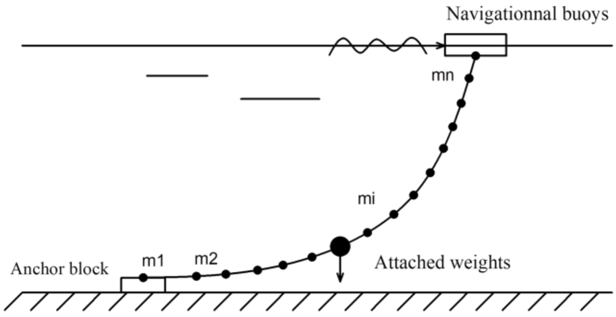

Fig. 1. Schematic modeling of mooring line.

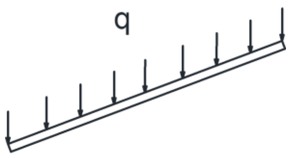

Fig. 2. Diagram of loads acting on the mooring line.

reviewing the past works, discussing the motivation of the recent study in Section 1, modeling of random wave is represented in Section 2. Finite element modeling of mooring lines with attached weights is discussed in Section 3. A case study is conducted in Section 4. Finally, conclusion is remarked in Section 5.

2. RANDOM WAVE MODELING

According to the theory of random waves, a random plane wave surface can be represented as the sum of harmonic waves with random phase angles. The random wave surface profile is determined by

$$\eta(x,t) = \sum_{i=1}^{N}[a_i \cos(k_i x - \omega_i t + \alpha_i)] \qquad (1)$$

where, a_i is amplitude, k_i is a wave number, and α_i denotes a random phase angle.

To determine the function η (random wave surface profile), it is necessary to identify the quantities a, k, and φ. If the random wave surface in the study area follows a certain spectrum, the expression is given by [10]

$$a_i = \sqrt{2S(\omega_i)\Delta\omega} \qquad (2)$$

where, $S(\omega_i)$ indicates frequency spectrum and $\Delta\omega$ denotes the frequency step.

The velocity component of random waves is expressed by

$$u(x,y,t) = \sum_{i=1}^{N}\left\{\begin{matrix}\frac{a_i g k_i}{\omega_i \cos(k_i d)}\cosh[k_i(y+d)] \\ \cos(k_i x - \omega_i t + \alpha_i)\end{matrix}\right\} \qquad (3)$$

When the wave spectrum at a specific sea area is known, the surface elevation and velocity of water particles in the x-direction can be determined. Commonly used spectra include the Pierson–Moskowitz (PM) spectrum and the JONSWAP spectrum. The PM spectrum is given by equation [10]

$$S_{PM}(\omega) = \frac{5}{16}H_S^2 \exp\left[-\frac{5}{4}\left(\frac{\omega}{\omega_p}\right)^{-4}\right] \qquad (4)$$

with $\omega_P = 2\pi/T_P$ being the peak frequency.

The JONSWAP spectrum is defined by equation [10]

$$S_J(\omega) = A_\gamma S_{PM}(\omega)\gamma^{\exp\left[-0,5\left(\frac{\omega-\omega_P}{\sigma\omega_P}\right)^2\right]} \qquad (5)$$

where, $S_{PM}(\omega)$ is PM spectrum function, γ is the dimensionless peak-shape parameter, and σ denotes the peak width parameter.

3. FINITE ELEMENT MODELING OF MOORING LINES WITH ATTACHED WEIGHTS

3.1. Mooring line modeling

Since mooring lines do not resist bending, they are modeled as a planar truss system using FEM. The system consists of bar elements connected by joints. Loads acting on the mooring line include its self-weight, attached masses, and buoy loads (from waves, wind, and currents), all applied at the nodes. The masses of the attached weights and elements are lumped at the nodes, as illustrated in Fig. 1.

3.2. Element stiffness matrix

According to [7], due to large nodal displacements, the mooring system is flexible. Thus, the element stiffness matrix is the sum of geometric and elastic stiffness matrices, as expressed in equation

$$[K]_e = [K_T]_e + [K_g]_e \qquad (6)$$

where, $[K_T]_e$ is the elastic stiffness matrix determined by the material properties, $[K_g]_e$ is the geometric stiffness matrix.

The geometric stiffness matrix is given by equation [7]

$$[K_g]_e = \frac{T}{L}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\end{bmatrix} \qquad (7)$$

The elastic stiffness matrix is given by equation [1]

$$[K]_e = \frac{EF}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad (8)$$

where, E is elastic modulus, F is cross-sectional area, L illustrates element length, finally T- indicates force that is dependent on displacement and unknown.

3.3. Element mass matrix

The element mass matrix is given by equation [1]

$$[M]_e = \frac{\rho FL}{6}\begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ 1 & 0 & 2 & 0\\ 0 & 1 & 0 & 2\end{bmatrix} \qquad (9)$$

with ρ being material density.

3.4. Element loads

The element is subjected to its self-weight acting vertically, as shown in Fig. 2. When the self-weight is assumed uniformly distributed, the nodal load vector is determined by equation

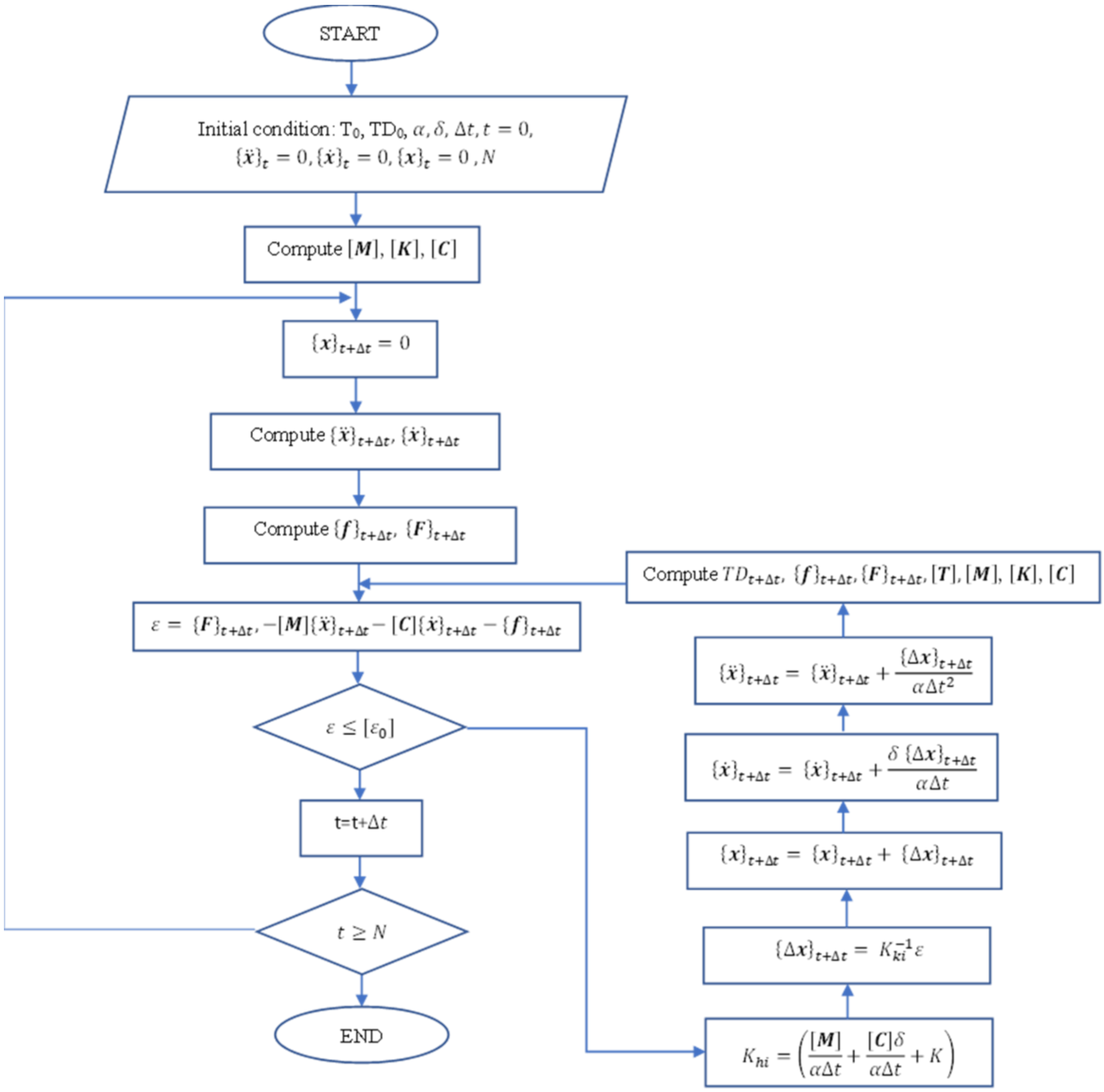

Fig. 3. Flow chart of Newmark method.

$$\{F\}_e = \begin{Bmatrix} 0 \\ -\frac{qL}{2} \\ 0 \\ -\frac{qL}{2} \end{Bmatrix} \quad (10)$$

with q being uniformly distributed load.

3.5. Coordinate transformation matrix

According to [1], the transformation matrix for a planar truss element is expressed as

$$[T]_e = \begin{bmatrix} l_x & m_x & 0 & 0 \\ l_y & m_y & 0 & 0 \\ 0 & 0 & l_x & m_x \\ 0 & 0 & l_y & m_y \end{bmatrix} \quad (11)$$

where, (l_x, m_x) and (l_y, m_y) are the direction cosines of xy in the global XY frame.

3.6. Loads from buoy and attached weights

The loads from attached weights and buoy are converted to nodal loads and added to the corresponding positions in the nodal force vector. The buoy is subjected to wind, current, and random wave loads as given by

$$F_p = F_g + F_{dc} + F_s \quad (12)$$

where, F_g is wind load, F_{dc} is current load, and F_s denotes wave load.

Wind and current loads are determined according to technical guidelines [9]. The wave load is determined similarly to the current load. Since the buoy draft is small, the water particle velocity is assumed uniform, given by

$$F_s = 0.59\rho_n A_p v(t)^2 \quad (13)$$

where, ρ_n denotes water density and A_p indicates buoy projected area. Meanwhile, $v(t)$ being water particle velocity at the surface y=0 is given by

$$v(t) = \sum_{i=1}^{N} \left[\frac{a_i g k_i}{\omega_i \cos(k_i d)} \cosh(k_i d) \cos(-\omega_i t + \alpha_i) \right] \quad (14)$$

3.7. Boundary conditions

Boundary conditions are applied at two ends of the mooring line: the anchor point and the buoy connection. At the anchor, both X and Y displacements are zero. At the buoy end, the Y displacement is constrained to zero.

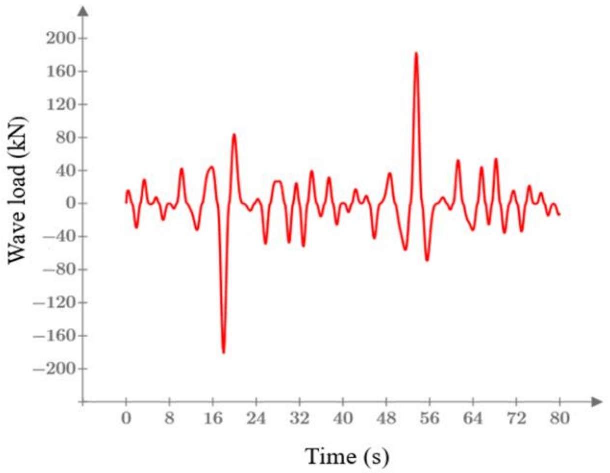

Fig. 4. Random wave load.

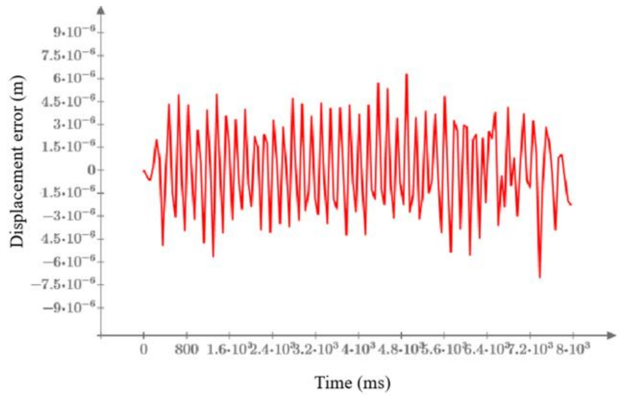

Fig. 5. Buoy displacement error.

3.8. Mooring line tension

The mooring line acts as a truss element under tension only. The tension force is computed from element elongation using equation

$$T = \frac{EF\Delta L}{L} \tag{15}$$

The tension is assumed constant in each element. From the tension, the nodal internal force vector is determined. Equilibrium is satisfied when the external nodal forces are equal to internal ones. The internal nodal force vector is given by

$$\{\boldsymbol{F_e}\}_{\boldsymbol{nl}} = \begin{Bmatrix} -T \\ 0 \\ 0 \\ T \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

3.9 Governing dynamic equations

With the stiffness, mass matrices, and nodal load vectors defined, the global system of equations for the mooring dynamics is given by [1]

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\} \tag{17}$$

where, $[K]$ is stiffness matrix, $[M]$ is mass matrix, $[C]$ denotes viscous damping matrix in which $[C] = a[M] + b[K]$, $\{F(t)\}$ is nodal load vector, $\{u\}$ illustrates displacement vector, $\{\dot{u}\}$ indicates velocity vector, and $\{\ddot{u}\}$ is acceleration vector.

3.10. Solution by Newmark method

The dynamic equation is solved using the direct time-integration Newmark method. The steps are as follows

(i) Determination of initial parameters

1. Input the characteristic data of the element, the initial nodal coordinates TD_0, the initial boundary conditions of displacement, acceleration, velocity $\{\ddot{x}\}_t = 0, \{\dot{x}\}_t = 0, \{x\}_t = 0$, calculation time N, and time step Δt.

2. Assume the initial tension force T_0.

3. Determine the initial element length vector: L_0, and the transformation matrix of the coordinate system $[T]_e$.

4. Determine the mass, stiffness, and viscous damping matrices $[M]$, $[K]$, and $[C]$ of the elements of the mooring system.

5. Determine the internal force vector at node $\{f\}$ based on the initial tension force.

(ii) Time loop

6. Assign displacement $\{x\}_{t+\Delta t} = 0$.

7. Compute acceleration $\{\ddot{x}\}_{t+\Delta t}$ using the formula:

$$\{\ddot{x}\}_{t+\Delta t} = \left\{\{x\}_{t+\Delta t} - \begin{bmatrix} \{x\}_t + \{\dot{x}\}_t\Delta t \\ +\{\ddot{x}\}_t\left(\frac{1}{2} - \alpha\right)\Delta t^2 \end{bmatrix}\right\}\frac{1}{\alpha\Delta t^2}.$$

8. Compute velocity $\{\dot{x}\}_{t+\Delta t}$ using the formula:

$$\{\dot{x}\}_{t+\Delta t} = \{\dot{x}\}_t + \{\ddot{x}\}_t(1-\delta)\Delta t + \{\ddot{x}\}_{t+\Delta t}\delta\Delta t.$$

9. Determine the nodal load vector: $\{\boldsymbol{F}\}_{\boldsymbol{t+\Delta t}}$.

10. Determine the residual vector of nodal forces:

$$\varepsilon = \{F\}_{t+\Delta t}, -[M]\{\ddot{x}\}_{t+\Delta t} - [C]\{\dot{x}\}_{t+\Delta t} - \{f\}_{t+\Delta t}.$$

(iii) Nodal equilibrium loop

11. Compute: $K_h = \left(\frac{[M]}{\alpha\Delta t} + \frac{[C]\delta}{\alpha\Delta t} + K\right)$.

12. Determine displacement increment:

$$\{\Delta \boldsymbol{x}\}_{t+\Delta t} = K_h^{-1}\varepsilon.$$

13. Determine updated displacement:

$$\{x\}_{t+\Delta t} = \{x\}_{t+\Delta t} + \{\Delta x\}_{t+\Delta t}.$$

14. Determine updated velocity:

$$\{\dot{x}\}_{t+\Delta t} = \{\dot{x}\}_{t+\Delta t} + \frac{\delta\,\{\Delta x\}_{t+\Delta t}}{\alpha\Delta t}.$$

15. Determine updated acceleration:

$$\{\ddot{x}\}_{t+\Delta t} = \{\ddot{x}\}_{t+\Delta t} + \frac{\{\Delta x\}_{t+\Delta t}}{\alpha\Delta t^2}.$$

16. Determine updated:

$$TD_{t+\Delta t}, \{f\}_{t+\Delta t}, \{F\}_{t+\Delta t}, [T], [M], [K], [C].$$

17. Recalculate the residual vector of nodal forces:

$$\varepsilon = \{F\}_{t+\Delta t}, -[M]\{\ddot{x}\}_{t+\Delta t} - [C]\{\dot{x}\}_{t+\Delta t} - \{f\}_{t+\Delta t}.$$

18. Check the nodal equilibrium condition$\varepsilon \leq [\varepsilon_0]$ if satisfied, exit the loop; if not, return to Step 11.

(iv) End of nodal equilibrium loop

19. Determine the reaction forces at the anchoring points.

20. Advance the time step t=t+Δt.

21. Check the total calculation time $t \geq N$. If not completed, return to Step 6; otherwise, terminate the time loop.

(v) End of time loop

The algorithms for Newmark method is illustrated in Fig. 3.

4. CASE STUDY

4.1. Initial data

A real navigational buoy was analyzed to demonstrate the method. The buoy has a diameter of 2.6 m, height 2.26 m, draft 1.03 m, connected by a 28.2 m mooring chain with 46 mm diameter. The water depth is 12.8 m. Three attached weights of 2 tons each are placed at 7 m intervals along the chain. Significant wave height H_S= 4.49 m, mean wave period T_m = 10 s. Wind load: 9.25 kN; current load: 0.57 kN. Wave load considered in two cases: quasi-static 76,396 kN for verification, and random waves for 80 s as shown in Fig. 4.

4.2 Program verification

To validate the algorithm and computational program, the buoy displacement and maximum mooring line tension were compared between the static case and the dynamic case with constant load using equation (17). In the static case [3], displacement and mooring tension are calculated from equation

$$[K]\{u\} = \{F\} \tag{18}$$

The total environmental load (wave, wind, current) on the buoy is 86.23 kN. After 80 s, the tension in both cases is 115.23 kN. The buoy displacement error is less than 10^{-5} m (Fig. 5). The comparison confirms the accuracy and reliability of the developed algorithm and computational tool.

4.3. Results

To calculate the displacement of the buoy and the tension in the mooring lines, the author conducted time-domain simulations over a period of 80 seconds. In order for the nodal equilibrium iteration to converge within the allowable error, the time step must be sufficiently small. Through testing with time steps of 0,1s; 0,01s and 0,001s, it was found that a time step of 0,001s provided good convergence. The calculation results are shown in Fig.s 6 and 7.

From the calculated time histories of displacement and mooring tension, the statistical characteristics of these quantities can be determined according to probability theory: maximum value $P_{\max}$, mean value μ, standard deviation σ, and cumulative frequency $F(x)$. These quantities are defined in Prime using the following functions

$$P_{max} = \max(X) \tag{19}$$

$$\mu = \text{mean}(X) \tag{20}$$

$$\sigma = \text{stedv}(X) \tag{21}$$

and

$$F(x) = \text{pnorm}(x, \mu, \sigma) \tag{22}$$

where, X is vector of tension forces at the connection point between the mooring line and the buoy and x is tension variable within the range $[0, P_{\max}]$.

Based on the design requirements, the mooring line tension corresponding to a given exceedance probability

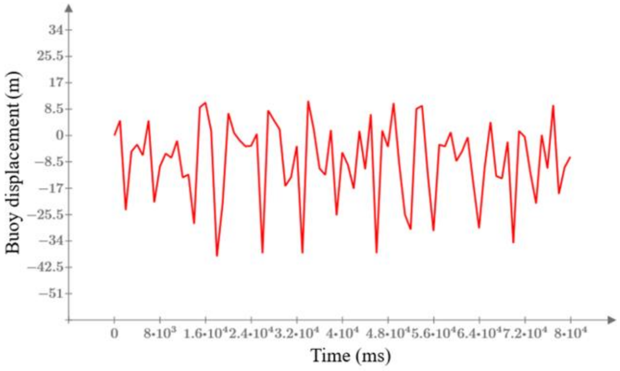

Fig. 6. Buoy displacement.

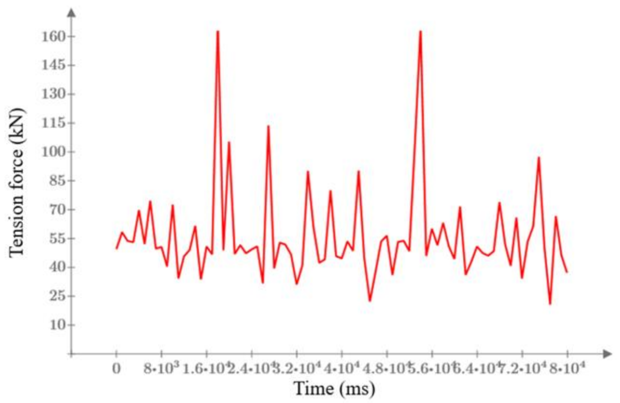

Fig. 7. Tension force at buoy connection point.

$i\%$ is determined in Prime by the function

$$P_{i\%} = \text{root}[F(x) - i\%, x, 0, P_{\max}] \tag{23}$$

5. CONCLUSION

This study presented the determination of random wave surface elevation and velocity from a known spectrum to calculate wave loads on the buoy. The FEM approach was used to model the mooring line as a planar truss system. The formulation of nodal load vectors, stiffness, mass, and transformation matrices was described. The Newmark method was employed to solve the dynamic equilibrium equations, and a computational program was developed. The program was applied to a real case study and validated, demonstrating its reliability. The research contributes a generalized and practical methodology for analyzing mooring lines with attached weights under stochastic environmental loading conditions.

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